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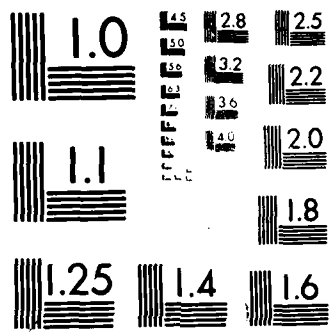

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The aim is to describe the exact shapes of objects that were meant to be circles or cylinders. The shapes are modeled as random fields whose parameter spaces are the intended shapes. A specific random field on a true circle is introduced via exponential smoothing of a random "noise," on the circle, with stationary and independent increments. The result is a stationary stochastically continuous random field. When the noise is the white noise, the result is the unique continuous stationary Gaussian random field on the circle. A similar definition for cylinders yields infinite dimentional Ornstein-Uhlenbeck processes.

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and

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RANDOM CIRCLES AND FIELDS ON CIRCLES*

by

E. ÇINLAR and J. G. WANG

1. INTRODUCTION

The overall aim is to describe the exact shapes of objects that were meant to be circles, cylinders, and so on. The motivation came from the need to model small deviations, from the intended surfaces, caused by random effects. For instance, in high precision gas lubricated bearings, the clearances are of the order of 10^{-5} inches and small discrepancies of the order of 10^{-6} and 10^{-7} must be taken into account for computing lubricant pressures and load capacities. Aside from such practical considerations, we are motivated by a desire to make precise the notion of a random circle, whose realizations would be called simply circles by the proverbial man in the street.

Analytically, we model the shapes as random fields whose parameter spaces are the intended shapes. Thus, for instance, a random circle is a random field on a true circle, which has enough stationarity and continuity to deserve the term circle.

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Preliminaries

Throughout, C will denote a circle of radius one, and m will be the length measure on it, that is, $m(A)$ is the length of A for each arc $A \subset C$. For $p, q \in C$, we write (p,q) for the open arc going from p to q counterclockwise, and we write $m(p,q)$ for the corresponding arc length. Note that, $(p,p) = C\setminus\{p\}$ and $m(p,p) = 2\pi$. The notations $[p,q]$, $(p,q]$, $[p,q]$, etc. are self-explanatory arcs. We take a point of C and distinguish it by labeling it 0. We associate the point $p \in C$ with the arc length $m(0,p)$, thus identifying C with $[0,2\pi]$, points 0 and 2π being the same. For $p,q \in C$, we define $p+q$ as the point whose numerical value is $m(0,p+q) = m(0,p) + m(0,q)$ modulo 2π. Finally, we write $-p$ for the point satisfying $(-p) + p = 0$.

Any function f on C can be regarded as a function on $\mathbb{R} = (-\infty,\infty)$ by identifying C with $[0,2\pi]$ and then extending f onto $\mathbb{R}$ by periodicity, i.e., by setting $f(2\pi n + x) = f(x)$ for all $0 < x \leq 2\pi$ and all integers n. Continuity, right-continuity, etc. for a function (and therefore a stochastic process) on C is to be understood in the usual sense for the periodic extension of the function onto $\mathbb{R}$.

Finally, throughout, $(\Omega, \mathcal{F}, P)$ will be the complete probability space that all probabilistic terms refer to.

Random fields and circles

A random field on C is a real-valued stochastic process having C as its parameter space. Each random field X on C can be thought as a stochastic process on $\mathbb{R}$ that is periodic with period 2π (that is, $X_{2\pi n+t} = X_t$ identically for all t and all integers n). With this identification, continuity, right-

continuity, stochastic continuity, stationarity, and reversibility carry over to random fields on C. We repeat some of these once more in the following.

(1.1) DEFINITION. Let $X = (X_p)_{p \in C}$ be a random field on C. Then,

a) X is said to be <u>stationary</u> if the probability law of $(X_{q+p})_{p \in C}$ is the same for all $q \in C$;

b) X is said to be <u>reversible</u> if the probability law of $(X_{-p})_{p \in C}$ is the same as that of X;

c) X is said to have the <u>Markov property</u> for the arc [p,q] if the collections $(X_r)_{r \in [p,q]}$ and $(X_r)_{r \in [q,p]}$ are conditionally independent given X_p and X_q; X is said to be <u>Markov</u> if it is Markov for every arc $[p,q] \subset C$.

(1.2) DEFINITION. A random circle is a positive, stationary, and stochastically continuous random field on C.

Each realization of a random field on C is a curve on the surface of a cylinder with base C. However, in the case of random circles at least, the point of view we adopt is that shown in the figure below, namely, each realization of a random circle is the boundary of a star-shaped set.

The stationarity refers to the invariance of the probability law under the rotations of C, or equivalently, under shifts of the "origin" O on C. The reversibility refers to the sameness of the probability law whether the figure is viewed from the front or the back of the page. In the absence of "past" and "future" on C, the Markov properly (1.1c) is the most appropriate,

and it is the most common way that property is defined for random fields. Namely, for each arc, the values in the interior and exterior of the arc are conditionally independent given the values at the two boundary points.

For a stochastic process defined on $\mathbb{R}$, CHAY (1972) used the phrase "quasi-Markov on $[0,T]$" to mean that for each sub-interval $[a,b]$ of $[0,T]$, the values of the process in the interior of $[a,b]$ and the exterior $[0,T]\setminus[a,b]$ of $[a,b]$ are conditionally independent given X_a, X_b. It is easy to show that the usual Markov property (of the past-future type) implies the quasi-Markov property on all of $\mathbb{R}$. Also, it is clear that every random field that is Markovian on C can be extended to a process on $\mathbb{R}$ that is quasi-Markov on $[0,2\pi]$.

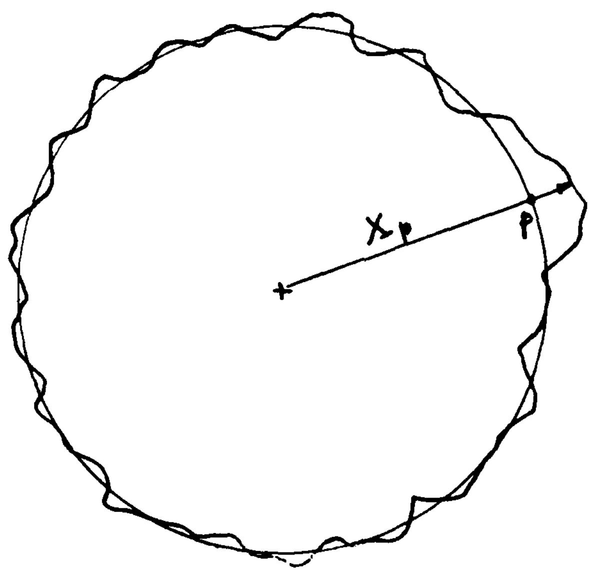

Figure. Point p is on the true circle C; X_p is the corresponding radius of the random circle

Examples and comments

Let W be a Wiener process on $\mathbb{R}_+$, let $a > 0$ be a fixed number, and define

$$(1.3) \qquad Y_t = W_{t+a} - W_t, \quad t \in \mathbb{R}_+; \qquad X_p = Y_p, \quad p \in C .$$

The process Y on $\mathbb{R}_+$ is continuous, stationary, Gaussian, and quasi-Markov (the latter property was noted first by SLEPIAN (1961)). Of course, X is Gaussian and Markovian on C. But, X is not continuous on C, having a jump at 0 with probability one, and of course, not stationary on C.

Stationarity on C is easy to achieve. It amounts to picking the distinguished point 0 at random on C. Let Y be an arbitrary random field on C, let U be independent of Y and have the uniform distribution on C, and set

$$(1.4) \qquad X_p = Y_{p-U}, \quad p \in C .$$

A quick computation with characteristic functions shows that, then, X is stationary on C. In particular, if Y is positive and continuous, then X is a continuous random circle. More particularly, let Y describe the largest square contained in C and having 0 as a corner; then, X is a continuous random circle according to Definition (1.2), even though every realization of X is a perfect square. This is somewhat unpleasant but seems inescapable. Our mental picture corresponds to (1.2) when the variance is small compared with the mean radius.

Let X be a Gaussian random field on C. In order for X to be stationary on C, we must have

(1.5) $$\mathrm{Cov}(X_p, X_q) = g \circ m(p,q) = g \circ m(q,p)$$

for some covariance function g. This implies, in particular, that every stationary Gaussian random field on C is reversible. For non-Gaussian fields, stationarity implies (1.5) but (1.5) does not imply reversibility (since covariances no longer determine the probability law). For instance, take Y in (1.4) to be increasing in the counterclockwise direction on $(0, 2\pi)$ and note that X is not reversible.

The form of stationary Gaussian random fields on C is well-known. LEVY (1951) has shown that every such field X with mean zero can be expressed as

(1.6) $$X_p = \sum_{n=0}^{\infty} c_n [U_n \cos np + V_n \sin np], \quad p \in C,$$

where the U_n and V_n are all independent Gaussian random variables with mean 0 and variance 1, and the constants c_n are square summable.

Interestingly enough, each term of the series (1.6) has the Markov property (1.1c) for most arcs. Consider

(1.7) $$X_p = U \cos np + V \sin np, \quad p \in C,$$

where the integer $n \geq 1$ is fixed, and U and V are independent, Gaussian, with mean 0 and variance 1. Clearly, X is continuous, stationary, Gaussian.

It has the Markov property for every arc [p,q] such that X_p and X_q determine U and V, which is whenever $\sin n(p-q) \neq 0$. Otherwise, if np-nq is a multiple of π, the Markov property for [p,q] fails. For instance, if $n = 1$ and $p = \pi/4$ and $q = 5\pi/4$, then $\cos np = \sin np$ and $\cos nq = \sin nq$, and X_p and X_q determine only the sum U + V, and it is easy to see that the Markov property fails. So, this X is <u>not</u> Markovian on C.

We shall show that the process X given by (1.6) with $c_o = 1/\sqrt{2}a$ $c_n = 1/\sqrt{a^2 + n^2}$ for some a is the unique Markovian field among all continuous, stationary, Gaussian ones, up to multiplication by and addition of constants.

<u>Integrators over C</u>

This is to introduce the notation and terminology we shall employ with stochastic integrals with respect to, essentially, processes with stationary and independent increments.

Throughout, W will denote the white noise and M an arbitrary <u>integrator</u> with stationary and independent increments. For every bounded Borel function f on C, we write

$$(1.8) \qquad Mf = \int_C f(p)\ M(dp)$$

for the stochastic integral of f with respect to M: this is the integral of f dZ over $(0,2\pi]$, where $dZ_p = M(dp)$ and Z is a process with stationary and independent increments. In the important special case where M = W, the corresponding Z is the Wiener process, which we denote by (W_p).

(1.9) REMARK. Independence of increments for M is equivalent to saying that $Mf_1,\ldots,Mf_n$ are independent whenever the f_k have disjoint supports. Stationarity for M is equivalent to saying that, for each f,

$$\int_C f(p)\ M(dp), \qquad \int_C f(p)\ M(q+dp)$$

have the same distribution for all $q \in C$. We express this by the phrase "the probability law of $M(q+\cdot)$ is free of q."

The structure of such M is well-known: for every bounded Borel function f on C,

$$(1.10) \qquad Mf = \alpha\ mf + \beta\ Wf + \int_{C\times\mathbb{R}} [N(dp,dz) - m(dp)n(dz)I_{(-1,1)}(z)]\ f(p)\ z,$$

where α and β are constants, m is the arc length measure and W is the white noise as mentioned before, and N is a Poisson random measure on $C\times\mathbb{R}$ with mean measure $m \times n$, the measure n being a Lévy measure on $\mathbb{R}$ (that is, n-integral of $z \longrightarrow z^2 \wedge 1$ is finite).

Organization

In the next section, we show that the random field

$$(1.11) \qquad X_p = \int_C e^{-am(p,q)}\ M(dq), \quad p \in C,$$

is stationary, right-continuous, stochastically continuous, and Markovian. In the special case M = W, it is further Gaussian, continuous, and reversible. Conversely, every continuous stationary Gaussian Markov random field on C has the form c + bX for some constants c and b.

If M is positive (which makes M into a positive compound Poisson random measure, basically), then X is positive and is a Markovian random circle. In the case $M = W$, X is not a random circle because it does take negative values; however, for small enough b and large enough c, $c + bX$ should be a good approximate model for a random circle with mean radius c.

In general, a whole class of Markovian random circles Y can be defined by letting $Y = f(X)$ for continuous, strictly increasing, strictly positive f.

In section 3 we concentrate on the special case of X defined by (1.11) with $M = W$, the white noise. The process X satisfies the same stochastic differential equation as does an Ornstein-Uhlenbeck process, but with an unusual boundary condition, namely, that the initial and final values must be the same. Therefore, the traditional definitions of a solution (with the initial value independent of the Wiener process) do not apply. We do obtain differential equations whose solutions in the traditional sense yield X. More interestingly, we give a decomposition

$$X_p = f(p)\, X_0 + Y_p, \qquad 0 \le p \le 2 ,$$

where Y is a Gaussian process independent of X_0 and is Markovian in the ordinary sense (past-future type). Finally, in the same section we give the Fourier representation for X.

Section 4 is devoted to extending X onto cylinders. In fact, this is not difficult. In the special case of Gaussian continuous fields on the cylinder $C \times \mathbb{R}$ we obtain an Ornstein-Uhlenbeck process (X_t) where each X_t is a random field $p \longrightarrow X_t(p)$ of the type (1.11).

The main difficulty in working on C is the absence of past and future. We have been able to avoid the issue by limiting ourselves to integrands that are deterministic. A general theory of stochastic integration on circles is yet to be developed. What we have here has the same relation to that future theory as does the Wiener integrals to Ito integrals.

2. STATIONARY MARKOV RANDOM FIELDS ON C

Throughout this section, M will denote an integrator on C with stationary and independent increments. The particular case where $M = W$, the white noise, is of special interest. Recall that m is the length measure on C. Define

$$(2.1) \qquad X_p = \int_C e^{-am(p,q)} M(dq), \quad p \in C,$$

where $a \geq 0$ is a constant. The case $a = 0$ is too trivial to be of interest but is useful in rounding out the theory. The following is the main result of this section.

(2.2) THEOREM

a) The random field X is stationary, right-continuous, stochastically continuous, and Markovian on C.

b) In the special case where $M = W$, X is continuous, Gaussian, and reversible in addition to being stationary and Markovian.

c) Conversely, every continuous, stationary, Gaussian, and Markovian random field Y on C has the form $Y = b + cX$, where b and c are constants and X is as in (2.1) with $M = W$.

If it were positive, X would be a Markovian random circle. This can be achieved by taking M positive, that is, by letting M have the form

$$(2.3) \qquad Mf = \alpha \cdot mf + \int_{C\times\mathbb{R}_+} N(dp,dz)\ f(p)z$$

with $\alpha \geq 0$ and the Poisson random measure N on $\mathbb{R}_+$ having the mean measure $m \times n$ such that the n-integral of $z \wedge 1$ is finite. We state this next.

(2.4) COROLLARY. Suppose M is positive. Then, X is a right-continuous Markovian random circle.

The discontinuities of X coincide with the atoms of M: for any $\omega\in\Omega$, $p \longrightarrow X_p(\omega)$ has a jump of size $-(1 - e^{-2\pi a})z$ at q *if and only if* the measure $N(\omega,\cdot)$ has an atom at the point (q,z). By taking the measure n vanishing outside an interval $I \subset \mathbb{R}$, one can limit the jumps of X to sizes $-(1 - e^{-2\pi a})z$ with $z\in I$. Thus, although a discontinuous *circle is repugnant, discontinuities* of magnitude 10^{-6} are not likely to cause comment in practice.

However, if continuity is needed, it can be achieved as follows. Let $f: \mathbb{R} \longrightarrow \mathbb{R}_+$ be a continuous strictly increasing function and define

$$(2.5) \qquad Y_p = f(X_p), \qquad p\in C .$$

(2.6) COROLLARY. The random field Y is a right-continuous Markovian random circle. In particular, if M = W, then Y is also continuous and reversible.

Even in the continuous case, by choosing f appropriately, one can limit the values of Y to any preselected positive open interval desired, and further, one can make the distribution of Y_p (necessarily free of p because of

stationarity) any continuous strictly increasing distribution function one desires: Choose $f = h \circ g$ where $g: \mathbb{R} \longrightarrow (0,1)$ is the (cumulative) Gaussian distribution function with mean 0 and variance that of X_p, and where h is the functional inverse of the distribution function desired for Y_p.

(2.7) CONJECTURE. Every continuous Markovian random circle Y has the form (2.5) where X is defined by (2.1) with M = W.

If true, this would be the equivalent, for Markov random fields on the circle, of FELLER's characterization of continuous Markov processes on $\mathbb{R}_+$. We are unable to prove this, but we suspect strongly that the conjecture is very near the truth.

The remainder of this section is devoted to proofs, basically.

Stationarity

This is easy to show for X. The following does it for a larger class.

(2.8) LEMMA. Let g be a bounded Borel function on $[0,2\pi]$ and define

$$(2.9) \qquad Z_p = \int_C g \circ m(p,q)\ M(dq), \quad p \in C .$$

Then, the random field Z is stationary on C.

PROOF. Fix $s \in C$, and note that

$$(2.10) \qquad Z_{s+p} = \int_C g\circ m(s+p,q)\ M(dq)$$

$$= \int_C g\circ m(s+p,\ s+q)\ M(s+dq) = \int_C g\circ m(p,q)\ M(s+dq),$$

that is, the random field $Z_{s+\cdot}$ is obtained from $M(s+\cdot)$ by the same rule as Z is from M. By the stationarity of M, $M(s+\cdot)$ has the same law as M. So, $Z_{s+\cdot}$ has the same law as Z.

The random field X has the form (2.9) with $g(x) = e^{-ax}$. Another special case is where $g(x) = x$ and $M = W$, in which case

$$(2.11) \qquad Z_p = Z_0 + 2\pi W_p - pW_{2\pi}, \qquad Z_0 = \int_0^{2\pi} s\ dW_s,$$

where (W_t) is a Wiener process, in other words, Z is a tied down Brownian motion with an initial value Z_0 that depends on the whole of W. This process was introduced by LEVY (1980, vol. 5, pp. 157 ff) who exploited its stationarity to handle the Wiener process by Fourier transforms.

Continuity

This is easier to discuss by first writing the integral (2.1) in the more traditional form. Identifying C with $(0,2\pi]$, separating the integral over C into integrals over $(0,p]$ and $(p,2\pi] = (0,2\pi]\setminus(0,p]$, and recalling that $m(p,q)$ is the length of the open arc (p,q) going from p to q counter-clockwise, we obtain, for $p\in(0,2\pi)$,

$$(2.12) \qquad X_p = \int_{(0,p]} e^{-a(2\pi-p+q)}\, M(dq) + \int_{(p,2\pi]} e^{-a(q-p)}\, M(dq)$$

$$= e^{ap}\Big(e^{-2\pi a} \int_{(0,p]} + \int_{(0,2\pi]} - \int_{(0,p]}\Big)\, e^{-aq}\, M(dq)$$

$$= e^{ap}\, X_0 - e^{ap}\,(1 - e^{-2\pi a}) \int_{(0,p]} e^{-aq}\, M(dq),$$

where, as in (2.1),

$$(2.13) \qquad X_0 = \int_{(0,2\pi]} e^{-aq}\, M(dq) \ .$$

The last integral in (2.12) is that of a continuous function with respect to a left-limited right-continuous process over $(0,p]$. It is now obvious that X is left-limited and right-continuous over $(0,2\pi)$. Taking limits as $p \downarrow 0$, we see that X is right-continuous at $p = 0$, and letting $p \uparrow 2\pi$, we see that X has a left-limit at $p = 0 = 2\pi$. So, X is left-limited and right-continuous on C.

It follows from (2.12) again that, for every outcome $\omega \in \Omega$, the function $p \longrightarrow X_p(\omega)$ has a jump of magnitude $-(1 - e^{-2\pi a})z$ at point q if and only if the measure $N(\omega,\cdot)$ has an atom (q,z).

Thus, $p \longrightarrow X_p(\omega)$ is continuous if and only if $N(\omega,\cdot)$ has no atoms. So, $p \longrightarrow X_p(\omega)$ is continuous for almost every ω if and only if the Lévy measure n vanishes, in which case the representation (1.10) for M becomes $M = bm + cW$ for some constants b and c. In particular, if $M = W$, X is continuous.

Finally, since the mean measure of N is $m \times n$ and m is diffuse, for each $p \in C$, the probability is zero that N has an atom on the ray $\{p\} \times \mathbb{R}$. Thus, for every fixed $p \in C$, X is continuous at p almost surely. In other words, X is stochastically continuous.

Markov property

For each arc $(p,q] \subset C$, let $\mathbf{F}_{pq}$ denote the σ-algebra generated by the increments $M(A)$, $A \subset [p,q]$, and let

$$(2.14) \qquad Z_{pq} = \int_{(p,q]} e^{-am(p,s)} M(ds) .$$

Independence of the increments of M implies that $\mathbf{F}_{pq}$ and $\mathbf{F}_{qp}$ are independent.

If $a = 0$ in (2.1), then $X_p = X_0$ for all p, and X is Markovian on C trivially. For the remainder of the proof we take $a \neq 0$.

For $p = q$, the Markov property for $[p,q]$ is trivially true. From here on, we fix $p,q \in C$, $p \neq q$.

Writing the integral (2.1) as the sum of two integrals, one over $(p,q]$ and the other over $(q,p]$, we obtain

$$(2.15) \qquad X_p = Z_{pq} + e^{-am(p,q)} Z_{qp}, \qquad X_q = e^{-am(q,p)} Z_{pq} + Z_{qp} .$$

Since $a \neq 0$, Z_{pq} and Z_{qp} are determined uniquely by X_p and X_q through these equations. Hence, the σ-algebra generated by X_p and X_q is the same as that generated by Z_{pq} and Z_{qp}.

Therefore, since $\mathbf{F}_{pq}$ and $\mathbf{F}_{qp}$ are independent, for every bounded $\mathbf{F}_{pq}$ - measurable random variable F,

$$E[F|X_p, X_q, \mathbf{F}_{qp}] = E[F|Z_{pq}, Z_{qp}, \mathbf{F}_{qp}] = E[F|Z_{pq}]$$

and

$$E[F|X_p,X_q] = E[F|Z_{pq}, Z_{qp}] = E[F|Z_{pq}],$$

which shows that

$$E[F|X_p, X_q, \mathbf{F}_{qp}] = E[F|X_p, X_q] \quad . \tag{2.16}$$

In other words, $\mathbf{F}_{pq}$ and $\mathbf{F}_{qp}$ are conditionally independent given X_p *and* X_q.

For $r \in (p,q)$, again separating the integral (2.1) for X_r into integrals over (p,q] and (q,p] and solving (2.15) for Z_{qp}, we obtain

$$\begin{aligned} X_r &= \int_{(p,q]} e^{-am(r,s)} M(ds) + e^{-am(r,q]} Z_{qp} \\ &= \int_{(p,q]} e^{-am(r,s)} M(ds) + \frac{1}{1-e^{-2\pi a}} (e^{-am(r,q)} X_q - e^{-am(r,p)} X_p). \end{aligned} \tag{2.17}$$

It is evident from this that

$$\sigma(X_r: r \in [p,q]) \subset \sigma(X_p, X_q, \mathbf{F}_{pq}) \tag{2.18}$$

and, by symmetry,

$$\sigma(X_r : r \notin [p,q]) \subset \sigma(X_p, X_q, \mathcal{F}_{qp}) . \tag{2.19}$$

In view of (2.16), namely, because the σ-algebras on the right sides of (2.18) and (2.19) are conditionally independent given X_p and X_q, we see that the desired Markov property holds for the arc $[p,q]$.

<u>Completion of the proof of Theorem (2.2)</u>

The preceding sub-sections have shown the truth of (a).

In the special case where $M = W$, we have shown that X is continuous on C. Obviously, in addition, X is Gaussian, and reversibility is immediate because every stationary Gaussian field on C is reversible. So, statement (b) holds.

To show (c), let Y be continuous, Gaussian, stationary, and Markovian on C. Subtracting a constant and dividing by another, we may assume that Y has mean 0 and variance 1. We need to show that the probability law of Y is that of bX for some constant b, which comes to showing that (since both X and Y are Gaussian),

$$EY_pY_q = b^2\, EX_pX_q\ , \qquad p \neq q\ . \tag{2.20}$$

In preparation for this we compute the covariance of X_p and X_q from (2.1) with $M = W$ and $a \neq 0$. We get, for $p \neq q$,

$$\begin{aligned} EX_pX_q &= E \int_C \int_C e^{-am(p,r)}\, e^{-am(q,s)}\, W(dr)\, W(ds) \\ &= \int_C e^{-am(p,r)-am(q,r)}\, m(dr) \end{aligned} \tag{2.21}$$

$$= \frac{1}{2a} (1 - e^{-2\pi a}) (e^{-am(p,q)} + e^{-am(q,p)}) ,$$

and

(2.22) $$E(X_p)^2 = \frac{1}{2a} (1 - e^{-4\pi a}) .$$

If $a = 0$, we have $X_p = W_{2\pi}$ for all p, and hence $EX_pX_q = 2\pi$. Thus, we need to show that, for some $a \geq 0$,

(2.23) $$EY_pY_q = (1 + e^{-2\pi a})^{-1} (e^{-am(p,q)} + e^{-am(q,p)}) ,$$

or equivalently, that the covariance function B, defined so that $EY_pY_q = B(m(p,q))$, has the form

(2.24) $$B(t) = be^{-at} + (1 - b)e^{at} , \quad b = (1 + e^{-2\pi a})^{-1} .$$

The remainder of the proof follows from a lemma of CHAY (1972). Suppose that Z is a stationary Gaussian process with mean 0 and variance 1 (stationarity over $\mathbb{R}$). Suppose that its covariance function is continuous. Suppose Z is quasi-Markov on [O,T]. Then, the covariance function f of Z satisfies $f''(t) = c \cdot f(t)$ for $t \leq T$. Depending on whether $c = 0$, $c < 0$, or $c > 0$, there are three possibilities for f(t):

(2.25) $$1 - \alpha t, \quad \cos \alpha t, \quad Ae^{-\alpha t} + (1 - A)e^{+\alpha t},$$

where $\alpha > 0$ in the first two cases and $\alpha \geq 0$ in the last case (and there are other conditions on α, A, T).

In the first case, the corresponding process has the form $W_{t+a} - W_t$ for $a = 1/\alpha$. This process is not continuous on C, hence $1 - \alpha t$ cannot be associated to Y.

In the second case, $\cos \alpha t$, the corresponding process has the form $U \cos \alpha t + V \sin \alpha t$. To be continuous on C, $\alpha > 0$ must be an integer. But, then, this becomes the random field (1.7), which we have already shown to fail the Markov property. So, the covariance function of Y cannot have the form $\cos \alpha t$.

The last possible form is in fact the form (2.24), because the condition of continuity for the field forces the values at $t = 0$ and $t = 2\pi$ to be the same, which condition becomes

$$1 = A e^{-2\pi\alpha} + (1 - A) e^{2\pi\alpha} , \tag{2.26}$$

which gives $A = b$ and $\alpha = a$ as in (2.24).

This completes the proof of Theorem (2.2).

Other remarks

Corollary (2.4) is immediate from Theorem (2.2a) and Definition (1.2) of random circle. The remarks on discontinuities of X following Corollary (2.4) were proved in the sub-section above entitled "continuity." Corollary (2.6) is immediate from Theorem (2.2) and the following facts. First, if Z is stationary, then so is f(Z) for any field Z and measurable f. Second,

similarly, if Z is reversible, so is f(Z). If Z is continuous and f is continuous, so is f(Z). If Z is Markovian and f is one-to-one, then f(Z) is Markovian, by the simple fact that, then, knowing $f(Z_p)$ is the same as knowing Z_p and vice-versa. When f is continuous and strictly increasing, f is obviously one-to-one from $\mathbb{R}$ onto some open interval.

Relationship to Ornstein-Uhlenbeck process

It is well-known that, when $\mathbb{R}_+$ is the parameter space, the only stationary, continuous, Gaussian Markov processes are Ornstein-Uhlenbeck processes. Since the random field X has the analogous properties on C when M = W, it is expected that X be the analog of the Ornstein-Uhlenbeck process on the circle C.

Indeed, the representation (2.12) shows this clearly. For greater clarity, we re-write (2.12) as a stochastic differential equation and with M = W:

$$dX_p = a\, X_p\, dp - b\, dW_p\,, \qquad 0 < p < 2\pi\,, \tag{2.27}$$

where $b = 1 - e^{-2\pi a}$. (Of course, b can be taken to be arbitrary; then, corresponding solution will be a multiple of X.) This is exactly the stochastic differential equation satisfied by Ornstein-Uhlenbeck processes. But, X is <u>not</u> an Ornstein-Uhlenbeck process, because the initial random variable X_0 depends on M = W through (2.13). Instead, X is the unique solution of (2.27) satisfying the two-sided boundary condition

$$X_0 = X_{2\pi} . \tag{2.28}$$

In the theory of stochastic differential equations, it has been customary (see IKEDA and WATANABE (1981) for instance) to make the convention that the initial value is independent of the driving process W or M, or whatever. On the circle, the initial value is also the final value, and such conventions will have to be altered.

Characteristic functional of X

We end this section with a formula: for any finite measure λ on C,

$$E \exp i \int_C \lambda(dp)\ X_p = \exp \int_C m(dq)\ g(\int_C \lambda(dp)\ e^{-am(p,q)}) \tag{2.29}$$

where g is the exponent function corresponding to M, that is, if M is represented by (1.10),

$$g(\theta) = i\alpha\theta - \frac{1}{2}\beta^2\theta^2 + \int_{\mathbb{R}} n(dz)\ [e^{i\theta z} - 1 - i\theta z\ I_{(-1,1)}(z)] . \tag{2.30}$$

Taking $\lambda = \lambda_0\,\delta_{p_0} + \ldots + \lambda_n\,\delta_{p_n}$, where $p_0, \ldots, p_n \in C$ are fixed and $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ are constants and where δ_p is the Dirac measure putting its unit mass at the point p, the left-side of (2.29) becomes the characteristic function of the random vector $(X_{p_0}, \ldots, X_{p_n})$. In particular, taking $\lambda = \lambda_o\,\delta_p$ with p fixed, we get

$$\begin{aligned} E \exp(i\,\lambda_0 X_p) &= \exp \int_C m(dq)\ g(\lambda_0\ e^{-am(p,q)}) \\ &= \exp \int_0^{2\pi} m(dq)\ g(\lambda_0\ e^{-aq}) . \end{aligned} \tag{2.31}$$

Two special cases of M deserve special mention: if $M = W$, the white noise, we have $g(\theta) = -\frac{1}{2}\theta^2$; and if M is positive like in (2.3), we have

$$g(\theta) = i\alpha\theta + \int_0^\infty n(dz)\ (e^{i\theta z} - 1)\ . \tag{2.32}$$

To show (2.29), first, we observe that

$$\int \lambda(dp)\ X_p = \int M(dq) \int \lambda(dp)\ e^{-am(p,q)} = \int M(dq)\ f(q) = Mf$$

with an obvious definition for f. Thus, it is sufficient to prove that

$$E\ e^{iMf} = e^{m(g\circ f)} \tag{2.33}$$

for every bounded Borel function f on C. Various continuity arguments reduce this to showing that (2.33) holds for f having the form $f = \theta_1 1_{A_1} + \ldots + \theta_n 1_{A_n}$ where $A_1, \ldots, A_n$ are disjoint arcs. But, then,

$$\begin{aligned} E\ e^{iMf} &= \prod_{k=1}^{n} \mathbf{E}(\exp i\theta_k\ M(A_k)) \\ &= \prod_{k=1}^{n} \exp(m(A_k)\ g(\theta_k)) = \exp \int m(dq)\ g(f(q)) \end{aligned}$$

as required.

3. STANDARD GAUSSIAN FIELD

This section is devoted to the random field X defined by (2.1) in the special case where M = W, that is,

$$(3.1) \qquad X_p = \int_C e^{-am(p,q)}\, W(dq), \quad p \in C .$$

We exclude the trivial case $a = 0$ and assume that $a > 0$. The random field X is stationary, reversible, continuous, Markovian, and Gaussian with mean 0 and covariance (see (2.21))

$$(3.2) \qquad B(p,q) = E\, X_p\, X_q = \frac{1}{2a}(1 - e^{-2\pi a})(e^{-am(p,q)} + e^{-am(q,p)}) .$$

The Markov property states that, given X_0 and X_p, the processes $(X_r\colon r \in [0,p])$ and $(X_r \in [p,2\pi])$ are conditionally independent given X_p. Thus, given X_0, the conditional law of X is the law of an ordinary (past-future type) Markov process. Our first aim is to identify that ordinary Markov process.

We had mentioned, in the last paragraphs of Section 2, that X is like an Ornstein-Uhlenbeck process, but it is not one because the stochastic differential equation to be solved is driven by a Wiener process W that depends on the initial value X_0. Our second aim is to obtain the equations for X whose driving term is independent of X_0.

Finally, we shall obtain the Fourier representation for X, which turns out to be almost as slow as that of the Wiener process.

Before listing the results, we introduce

$$b = 1 - e^{-2\pi a}; \quad m_{pq} = a\, m(p,q), \quad p,q \in C, \tag{3.3}$$

in order to lighten the notation somewhat. We shall be using hyperbolic functions extensively.

(3.4) THEOREM. Let $[p,q]$ be a fixed closed arc on C with $0 \leq p < q \leq 2\pi$. There exists a Wiener process $\hat{W}$ on $[p,q]$ independent of X_p and X_q such that

$$X_r = \frac{\sinh m_{rq}}{\sinh m_{pq}} X_p + \frac{\sinh m_{pr}}{\sinh m_{pq}} X_q + b \int_{[p,r]} \frac{\sinh m_{rq}}{\sinh m_{sq}} \hat{W}(ds) \tag{3.5}$$

for all $r \in [p,q]$.

(3.6) COROLLARY. Let Y_r be the last term on the right side of (3.5). Considered as a process on the time interval $[p,q]$, Y is a Markov process (in the ordinary sense), is independent of X_p and X_q, and has $Y_p = Y_q = 0$. Moreover, it satisfies the stochastic differential equation

$$dY_r = -a(\operatorname{ctnh} m_{rq})\, Y_r\, dr + b\hat{W}(dr), \quad p < r < q. \tag{3.7}$$

(3.8) COROLLARY. Given X_p and X_q, the conditional law of $(X_r)_{r\in[p,q]}$ is the law of an ordinary Markov process on $[p,q]$. The process satisfies the stochastic differential equation

(3.9) $dX_r = (aX_q \operatorname{csch} m_{rq} - aX_r \operatorname{ctnh} m_{rq})dr + b\hat{W}(dr), \quad p < r < q ,$

where $\hat{W}$ is a Wiener process independent of the pair (X_p, X_q).

(3.10) PROPOSITION. There exists a Wiener process $\hat{W}$ independent of X_0 such that

$$(3.11) \qquad X_p = \frac{\cosh(\pi a - pa)}{\cosh \pi a} X_0 + b \int_{[0,p]} \frac{\sinh m_{p,2\pi}}{\sinh m_{s,2\pi}} \hat{W}(ds), \qquad p \in C .$$

Thus, given X_0, the conditional law of X is the law of an ordinary Markov process on the time interval $[0,2\pi]$.

(3.12) REMARK. Still other facts about X can be obtained from Corollaries (3.6) and (3.8) by replacing p and q there with 0 and 2π respectively.

(3.13) THEOREM. The random field X has the series representation

$$(3.14) \qquad X_p = \frac{b}{a\sqrt{2\pi}} U_0 + \frac{b}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{a^2+n^2}} [U_n \cos np + V_n \sin np] ,$$

where $U_0, U_1, \ldots, V_1, V_2, \ldots$ are independent Gaussian random variables with mean 0 and variance 1. In fact $U_0 = W_{2\pi}/\sqrt{2\pi}$, and for $n \neq 0$,

$$(3.15) \qquad U_n = \frac{1}{\sqrt{\pi(a^2+n^2)}} \int_0^{2\pi} [a \cos nq + n \sin nq] \, W(dq) ,$$

$$V_n = \frac{1}{\sqrt{\pi(a^2+n^2)}} \int_0^{2\pi} [a \sin nq - n \cos nq] \, W(dq) .$$

The remainder of this section is devoted to proofs. Corollaries (3.6) and (3.8) are easy consequences of Theorem (3.4). Proposition (3.10) is obtained from Theorem (3.4) and Corollary (3.8) by replacing p with 0 and q with 2π. Thus, the only things to be proved are (3.4), the Markov representation theorem, and (3.13), the Fourier series representation.

Proof of Theorem (3.4)

For $r \in [p,q]$, define

$$(3.17) \qquad Y_r = X_r - \frac{\sinh m_{rq}}{\sinh m_{pq}} X_p - \frac{\sinh m_{pr}}{\sinh m_{pq}} X_q ,$$

$$(3.18) \qquad \hat{W}_r = \frac{1}{b} Y_r + \frac{a}{b} \int_{[p,r]} Y_u \operatorname{ctnh} m_{uq} \, du .$$

It is clear that Y is a continuous Gaussian process on [p,q] with mean 0 and $Y_p = Y_q = 0$. So, $\hat{W}$ is again a continuous Gaussian process with mean 0, and $\hat{W}_p = 0$. In differential form, (3.18) gives

$$d\left(\frac{1}{\sinh m_{rq}} Y_r\right) = \frac{1}{\sinh m_{rq}} \left(dY_r + \frac{\cosh m_{rq}}{\sinh m_{rq}} a \, Y_r \, dr\right)$$

$$= \frac{b}{\sinh m_{rq}} d\hat{W}_r .$$

Hence, since $Y_p = 0$,

$$Y_r = \sinh m_{rq} \int_{[p,r]} \frac{b}{\sinh m_{sq}} d\hat{W}_s .$$

Putting this into (3.17) and rearranging the terms yield (3.5).

To complete the proof, we need to show that $\hat{W}$ is a Wiener process and is independent of X_p and X_q. Since all the processes are Gaussian, these matters are reduced to computing various covariances.

First we express (3.2) in different notation:

$$E\, X_r X_s = \frac{b}{a} e^{-\pi a} \cosh (\pi a - m_{rs}) , \quad r,s \in C . \tag{3.19}$$

Using this with (3.17) and some elementary identities yields

$$E\, Y_r X_p = E\, Y_r X_q = 0 , \qquad r \in [p,q] , \tag{3.20}$$

$$E\, Y_r Y_s = \frac{b^2}{a \sinh m_{pq}} \sinh m_{pr} \sinh m_{sq} , \quad p \leq r < s \leq q . \tag{3.21}$$

Since X and Y are Gaussian, (3.20) implies the independence of the process Y from the pair (X_p, X_q), and in view of (3.18), the process $\hat{W}$ is independent of X_p and X_q.

Since $\hat{W}_p = 0$ and since $\hat{W}$ is a continuous Gaussian process with mean 0, in order to show that $\hat{W}$ is a Wiener process, it is sufficient to show that

$$E\, \hat{W}_r \hat{W}_s = r - p , \qquad p \leq r < s \leq q . \tag{3.22}$$

This computation requires some delicacy, for which reason we provide a few details.

It turns out to be useful to introduce the notations

$$F_u = \int_{[p,u]} \sinh m_{pv} \operatorname{ctnh} m_{vq}\, dv , \tag{3.23}$$

$$(3.24) \qquad aG_u = \int_{[p,u]} a \cosh m_{vq}\, dv = \sinh m_{pq} - \sinh m_{uq} .$$

It follows from (3.18) that, for $r < s$,

$$b^2 \hat{W}_r \hat{W}_s = Y_r Y_s + a \int_p^r du\, Y_u Y_s \operatorname{ctnh} m_{uq}$$

$$+ a(\int_p^r du + \int_r^s du)\, Y_u Y_r \operatorname{ctnh} m_{uq}$$

$$+ a^2 \int_p^r du\, (\int_p^u dv + \int_u^s dv)\, Y_u Y_v \operatorname{ctnh} m_{uq} \operatorname{ctnh} m_{vq}$$

from which, taking expectations with the aid of (3.21) and using the notations of (3.23) and (3.24), we obtain

$$(3.25) \qquad a \sinh m_{pq}\, E\, \hat{W}_r \hat{W}_s$$

$$= \sinh m_{pr} \sinh m_{sq} + a \sinh m_{sq} \int_p^r dF_u$$

$$+ a \sinh m_{rq} \int_p^r dF_u + a \sinh m_{pr} \int_r^s dG_u$$

$$+ a^2 \int_p^r dG_u \int_p^u dF_v + a^2 \int_p^r dF_u \int_u^s dG_v$$

$$= \sinh m_{pr} [\sinh m_{sq} + a(G_s - F_r)]$$

$$+ a[\sinh m_{sq} + \sinh m_{rq} + aG_r + aG_s]\, F_r - 2a^2 \int_p^r G_u dF_u .$$

Note that, directly from (3.23) and (3.24),

$$2a^2 \int_p^r G_u \, dF_u = 2aF_r \sinh m_{pq} - 2a \int_p^r \sinh m_{pu} \cosh m_{uq} \, du$$

$$= 2aF_r \sinh m_{pq} - a \int_p^r [\sinh m_{pq} + \sinh(m_{pu} - m_{uq})] du$$

$$= 2aF_r \sinh m_{pq} - a \cdot (r - p) \sinh m_{pq}$$

$$- \frac{1}{2} \cosh(m_{pr} - m_{rq}) + \frac{1}{2} \cosh m_{pq} .$$

Putting this into (3.25) and simplifying, we obtain (3.22), thus completing the proof of Theorem (3.4).

Proof of Theorem (3.13)

Considering X as complex valued, Fourier series for X is

(3.26) $$X_p = \sum_{n=-\infty}^{\infty} A_n e^{inp} , \quad p \in C ,$$

where

(3.27) $$A_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-inp} X_p \, dp .$$

It will be convenient to introduce, in addition to $b = 1 - e^{-2\pi a}$,

(3.28) $$c_0 = \frac{1}{a\sqrt{2\pi}} ; \quad c_n = \frac{1}{\sqrt{\pi(a^2+n^2)}} , \quad n \neq 0 .$$

Now,

$$A_n = \frac{1}{2\pi}\int_0^{2\pi} dp \cdot e^{-inp} \int_C W(dq)\; e^{-am(p,q)}$$

$$= \frac{1}{2\pi}\int_C W(dq)\; \frac{b}{a-in}\; e^{-inq}$$

$$= \frac{b}{2\pi(a^2+n^2)} \int_C W(dq)(a + in)(\cos nq - i \sin nq)\ .$$

Putting $U_0 = W_{2\pi}/\sqrt{2\pi}$ and defining U_n and V_n for $n \neq 0$, for n positive and negative, by (3.15) and (3.16), we see that

$$A_0 = bc_0U_0\ ; \qquad A_n = \frac{1}{2}\, bc_n\ (U_n - iV_n)\ , \qquad n \neq 0\ .$$

Putting these into (3.26) yields

$$(3.29) \qquad X_p = bc_0U_0 + \frac{b}{2}\sum_{n\neq 0} c_n(U_n - iV_n)(\cos np + i \sin np)$$

$$= bc_0U_0 + \frac{b}{2}\sum_{n=1}^{\infty} c_n[(U_n + U_{-n} - iV_n - iV_{-n}) \cos np$$

$$+ (iU_n - iU_{-n} + V_n - V_{-n}) \sin np]$$

$$= bc_0U_0 + \frac{b}{2}\sum_{n=1}^{\infty} c_n(2U_n \cos np + 2V_n \sin np)$$

since $U_{-n} = U_n$ and $V_{-n} = -V_n$ for all $n \geq 1$. This is exactly (3.14), and the only thing remaining is to show the claims about the nature of U_n and V_n.

It is obvious that they are Gaussian and have mean 0. The remaining assertions about their variances and independence from each other are proved by showing that

$$E\,U_n^2 = E\,V_n^2 = 1\,, \quad E\,U_n U_m = E\,V_n V_m = 0\,, \quad n \neq m\,,$$

and that

$$E\,U_n\,V_m = 0\,, \quad n \geq 0\,, m > 0\,.$$

This requires only elementary calculus once we note that

$$E \int_C f(p)\;W(dp) \int_C g(q)\;W(dq) = \int_0^{2\pi} f(p)\;g(p)\;dp\,.$$

(3.30) REMARK. One could use the fact that X is stationary and periodic and $EX_pX_q = B(p - q)$ where

$$B(t) = \frac{b^2}{2\pi a^2} + \sum_{n=1}^{\infty} b^2\,c_n^2 \cos nt\,, \quad t \in \mathbb{R}\,,$$

and then use the spectral decomposition of stationary Gaussian processes to arrive at (3.14). The proof above is longer, but it identifies the coefficients U_n and V_n.

4. FIELDS ON CYLINDERS

Extending the results of the preceding sections to random fields defined on cylinders presents no new difficulties. We limit ourselves, therefore, to introducing the random field of interest and to stating several interesting properties.

Let C be the circle of radius one as before and consider the cylinder $C\times\mathbb{R}$. For p and q on C, we write m(p,q) for the length of the arc (p,q) going from p to q counterclockwise, just as before. In addition, we write m(s,t) for the length of the interval $(s,t) \subset \mathbb{R}$, although not permissible by the strict rules for notation making.

Let M be an integrator on $C \times \mathbb{R}$ with stationary and independent increments: for every bounded Borel function f with compact support in $C \times \mathbb{R}$,

$$
(4.1) \qquad Mf = \alpha \int_{C\times\mathbb{R}} m(dp)\, dt\, f(p,t) + \beta \int_{C\times\mathbb{R}} W(dp,dt)\, f(p,t)
$$

$$
+ \int_{C\times\mathbb{R}\times\mathbb{R}} [N(dp,dt,dz) - m(dp)dtn(dz)I_{(-1,1)}(z)]\, f(p,t)z\,,
$$

where α and β are constants, N is a Poisson random measure on $C\times\mathbb{R}\times\mathbb{R}$ with mean measure element m(dp) dt n(dz), where n is a Lévy measure, and W is the white noise on $C\times\mathbb{R}$. The last may be thought as the integrator corresponding to Brownian sheet, that is,

$$
(4.2) \qquad W_{p,t} = \int_{(0,p]\times(0,t]} W(dq,ds)\,, \quad 0 \le p \le 2\pi,\ 0 \le t < \infty\,,
$$

is a Brownian sheet on $[0,2\pi] \times \mathbb{R}_+$, which is a Gaussian process with mean 0 and

(4.3) $$E\, W_{p,t}\, W_{q,u} = (p \wedge q)(t \wedge u) \ .$$

Integration with respect to W was considered by WONG and ZAKAI (1974), CAIROLI and WALSH (1975), and others. The last integral in (4.1) is in fact simpler and should be understood in analogy with the simple time case. The integral over $C \times \mathbb{R} \times (R \setminus (-1,1)) = E$ is that of $f(p,t)z$ with respect to a random measure, and hence, is an ordinary integral. The integral on the complement of E is a stochastic integral; it is the limit in probability (indeed almost sure limit) of

$$\int_{E_\varepsilon} [N(dp,dt,dz) - m(dp)\ dt\ n(dz)]\ f(p,t)z \ ,$$

as $\varepsilon \downarrow 0$, where $E_\varepsilon = C \times \mathbb{R} \times ((-1,1) \setminus (-\varepsilon,\varepsilon))$.

The random field of interest is

(4.4) $$X(p,t) = \int_{C \times (-\infty,t]} e^{-am(p,q)-bm(s,t)}\ M(dq,ds), \qquad p \in C,\ t \in \mathbb{R},$$

where $a \geq 0$ and $b > 0$ are fixed constants. We view it as a random field on the cylinder $C \times \mathbb{R}$. Alternately, we may regard it as the evolution in time of a random field on the circle C.

The message of the following theorem is that the sections of X are all stationary and Markov. Recall that a time-homogeneous Markov process, with time-set $\mathbb{R}_+$ and taking values in a Lusin space, is called a <u>Hunt process</u> if it is right-continuous, left-limited, strong Markov, and quasi-left-continuous, the last two properties being with respect to a filtration that is right-continuous and augmented properly. See BLUMENTHAL and GETOOR (1968) for the definitions. We shall use the term here for processes with time-set $\mathbb{R}$, without specifying the filtration, and somewhat fraudulently: Extension of the concepts onto $\mathbb{R}$ is no problem; we may take as filtration the $(\mathbb{F}_t)_{t\in\mathbb{R}}$, where $\mathbb{F}_t = \mathbb{F}_t^o \vee \mathbb{N}$, $\mathbb{F}_t^o$ being the σ-algebra generated by $\{Mf: f$ continuous with compact support contained in $C \times (-\infty,t]\}$ and $\mathbb{N}$ being all the null-sets of the completion of $\mathbb{F}_\infty^o$; the fraud is that we have only one probability measure P, instead of a whole collection, one for each starting state, of which we have none.

(4.5) THEOREM. a) For each $t \in \mathbb{R}$, $p \longrightarrow X(p,t)$ is a stationary, right-continuous stochastically continuous, Markovian random field on C; the law of $p \longrightarrow X(p,t)$ is free of t.

b) For each $p \in C$, $t \longrightarrow X(p,t)$ is a stationary Hunt process.

c) For any integer $n \geq 1$ and points $p_1,\dots,p_n$ on C, the process $t \longrightarrow (X(p_1,t),\dots,X(p_n,t))$ is a stationary Hunt process with state space $\mathbb{R}^n$.

d) Let $X(\cdot,t)$ denote the mapping $p \longrightarrow X(p,t)$. Then, the process $t \longrightarrow X(\cdot,t)$ is a stationary Hunt process taking values in the space $\mathbb{E}$ of all right-continuous, and left-limited functions from C into $\mathbb{R}$, topologized by uniform convergence on C.

(4.6) COROLLARY. Suppose that $M = W$, the white noise on $C \times \mathbb{R}$. Then, in addition to the properties above, X is a continuous stationary Gaussian random field on $C \times \mathbb{R}$ with mean 0 and

$$E\, X(p,t)\, X(q,u) = \frac{1}{2b} B(p,q)\, e^{-b|u-t|}, \quad p,q \in C, \quad t,u \in \mathbb{R}. \tag{4.7}$$

where B is as given by (3.2). Moreover,

a) for each $p \in C$, $t \longrightarrow X(p,t)$ is an Ornstein-Uhlenbeck process;

b) for any integer $n \geq 1$ and points $p_1,\dots,p_n \in C$, the process $t \longrightarrow (X(p_1,t),\dots,X(p_n,t))$ is an $\mathbb{R}^n$ - valued Ornstein-Uhlenbeck process, which is a weak solution of

$$dX^i(t) = -bX^i(t)\, dt + \sum_{j=1}^{n} c_{ij} W^j(dt), \tag{4.8}$$

where $X^i(t) = X(p_i,t)$, and $W^1,\dots,W^n$ are independent Wiener processes, and the matrix c satisfies $cc' = B$, that is,

$$\sum_{k=1}^{n} c_{ik}\, c_{jk} = B(p_i,p_j), \qquad 1 \leq i,j \leq n; \tag{4.9}$$

c) the process $t \longrightarrow X(\cdot,t)$ is an Ornstein-Uhlenbeck process taking values in the space $\mathbb{C}$ of all continuous functions from C to $\mathbb{R}$, topologized by uniform convergence on C.

(4.10) REMARK. An Ornstein-Uhlenbeck process with values in $\mathbb{R}$ or $\mathbb{R}^n$ is a process that satisfies an equation like (4.8), see IKEDA and WATANABE (1981) for instance. In the case of processes taking values in infinite dimensional

spaces like $\mathbb{C}$ in (4.6c), see ITO (1982) and MEYER (1982) for definitions. Our case is much clearer, because of the explicit formula (4.4).

(4.11) THEOREM. The random field X enjoys the following Markov property. Let $A \subset C$ be an arc, let $B \subset \mathbb{R}$ be an interval, and consider the rectangle $A \times B$ on the surface of the cylinder $C \times \mathbb{R}$. Then, $\{X(p,t): \ p \in A \text{ and } t \in B\}$ and $\{X(p,t): \ p \notin A \text{ or } t \notin B\}$ are conditionally independent given $\{X(p,t): (p,t) \in \partial(A \times B)\}$, $\partial(A \times B)$ being the boundary of $A \times B$.

(4.12) REMARK. The arc A can be taken to be C, in which case $A \times B$ is a cylinder, and the Markov property above coincides with the statement (4.5d). The degenerate arc $A = \{p\}$ is allowed. Similarly, B does not have to be an interval.

(4.13) REMARK. By limiting the parameter t to an interval $D \subset \mathbb{R}$, we may regard X as a field on the cylinder $C \times D$. For this purpose, $D = \mathbb{R}_+ = [0,\infty)$ and $D = [0,d]$, $d > 0$, are of special interest. All the results listed above continue to hold for such an X. In considering such cylinders $C \times D$, it may appear more natural to define X via integrators on $C \times D$. Assuming $D = [0,d]$ or $D = [0,\infty)$, this can be done by setting

$$(4.14) \qquad X(p,t) = e^{-bt} X(p,0) + \int_{C \times (0,t]} e^{-am(p,q)-bm(s,t)} M(dq,ds),$$

where

$$(4.15) \qquad X(p,0) = \int_C e^{-am(p,q)} M_0(dq) ,$$

M_0 being an integrator on C independent of the integrator M on $C \times D$. Of course,

$$(4.16) \qquad M_0(dq) = \int_{-\infty}^{0} e^{bs}\, M(dq,ds) \;.$$

The formula (4.14) can be used to define an "Ornstein-Uhlenbeck" process $t \longrightarrow X(\cdot,t)$ on $\mathbb{R}_+$ and taking values in $\mathbb{E}$ of (4.5d) with initial state $X(\cdot,0)$ arbitrary and independent of M on $C \times \mathbb{R}_+$. In an obvious sense, the probability law of $X(\cdot,t)$ approaches, as $t \longrightarrow \infty$, to that of the right-side of (4.15) with M_0 having the same law as the right-side of (4.16).

Many of the results above rest on, or motivated by, the following observation. Its truth is immediate from Remark (1.9) and a similar statement for integrators on $\mathbb{R}$.

(4.17) LEMMA. Define $M(g \times h)$ by (4.1) with $f(p,t) = g(p)\, h(t)$, where g is a bounded Borel function on C and h is bounded Borel function with compact support in $\mathbb{R}$. Then,

a) for fixed g, $h \longrightarrow M(g \times h)$ is an integrator on $\mathbb{R}$ with stationary and independent increments;

b) for fixed h, $g \longrightarrow M(g \times h)$ is an integrator on C with stationary and independent increments.

In a similar vein, and more interesting, is the following "stochastic kernel." For h bounded, Borel, with compact support in $\mathbb{R}$, define

$$K(p,h) = \int_{C \times \mathbb{R}} e^{-am(p,q)} \cdot h(t)\ M(dq,dt) \ . \tag{4.18}$$

Recall that $\mathbb{E}$ stands for the space of all right-continuous left-limited functions from C into $\mathbb{R}$ (see (4.5d)). The following is immediate from the preceding lemma.

(4.19) PROPOSITION. For each fixed h, $p \longrightarrow K(p,h)$ is a random element of $\mathbb{E}$. For each fixed p, $h \longrightarrow K(p,h)$ is an integrator on $\mathbb{R}$ with stationary and independent increments.

Proof of Theorem (4.5)

Stationarity of the increments of M means that the probability law of $M(p + dq,\ t + du)$ is free of (p,t). This implies that the law of X is free of the choice of origin and of rotations of the coordinate system. This shows the stationarity in all the statements of (4.5).

For fixed $t \in \mathbb{R}$, let $h(s) = e^{-b(t-s)}$ for $s \leq t$ and $h(s) = 0$ for $s > t$. Then, by Lemma (4.17b), $g \longrightarrow M(g \times h)$ is an integrator on C with stationary and independent increments, and $p \longrightarrow X(p,t)$ is defined from it according to the formula (2.1). Thus, (4.5a) is a re-statement of Theorem (2.2a).

For fixed $p \in C$, let $g(q) = e^{-am(p,q)}$. By Lemma (4.17a), $h \longrightarrow M_p(h) = M(g \times h)$ is an integrator on $\mathbb{R}$ with stationary and independent increments, and we have

$$X(p,t) = \int_{(-\infty,t]} e^{-bm(s,t)}\ M_p(ds) \ . \tag{4.20}$$

That $t \longrightarrow X(p,t)$ is left-limited and right continuous is immediate from this. We have

$$(4.21) \qquad X(p,t+u) = e^{-bu} X(p,t) + \int_0^u e^{-bm(s,u)} M_p(t+ds) .$$

Since $M_p(t+\cdot)$ is independent of the past $\mathbb{F}_t$ and has the same law as $M_p(\cdot)$, (4.21) implies that $t \longrightarrow X(p,t)$ is a time-homogeneous Markov process. Strong Markov property is similar; one can replace t by a stopping time T in (4.21) and use the known "strong Markov property" for M_o. Finally, quasi-left continuity comes from the facts that every jump of $t \longrightarrow X(p,t)$ coincides with an atom (t,z) of the Poisson random measure involved with M_o and therefore, the jump times are exhausted by a sequence of stopping times each one of which is totally unpredictable. This completes the proof of (4.5b).

The proofs of (4.5c,d) are entirely similar: To show (4.5d), replace M_o in the preceding paragraph with $h \longrightarrow K(\cdot,h)$, the latter being defined by (4.18) and having the properties mentioned in Proposition (4.19).

<u>Proof of Corollary (4.6)</u>

When $M = W$, the white noise on $C \times \mathbb{R}$, it is obvious that X becomes Gaussian, stationary, reversible, continuous. It is easy to compute the covariance function and show that (4.7) is indeed true.

The remaining statements are all similar. We prove (b) only. First, putting $M = W$ and taking differentials in (4.20) we have

$$(4.22) \qquad dX(p,t) = -bX(p,t)dt + W_p(dt)$$

where

$$W_p(h) = \int_{C \times \mathbb{R}} e^{-am(p,q)} \, h(t) \, W(dq,dt) \; . \tag{4.23}$$

Let $p_1,\ldots,p_n \in C$ be fixed. For $h_1,\ldots,h_n$ bounded, Borel, with bounded support, we have

$$E\, W_{p_i}(h_i)\, W_{p_j}(h_j) =$$

$$= \int_{C \times \mathbb{R}} \exp[-am(p_i,q) - am(p_j,q)]\, h_i(t)\, h_j(t)\, m(dq)dt$$

$$= B(p_i,p_j) \int_{\mathbb{R}} h_i(t)\, h_j(t)\, dt$$

$$= E\Big(\sum_{k=1}^{n} c_{ik}\, W^k(h_i)\Big)\Big(\sum_{k=1}^{n} c_{jk}\, W^k(h_j)\Big) \; ,$$

with (c_{ij}) and $W^j)$ as described in (4.6b). Thus, the probability law of the n-dimensional process $(W_{p_1},\ldots,W_{p_n})$ is that of $\Big(\sum_{k=1}^{n} c_{ik}\, W^k\Big)_{i=1,\ldots,n}$. Putting this fact together with (4.22) we obtain (4.8) and the proof of (4.6b) is complete.

Proof of the Markov property (4.11)

It is sufficient and convenient to give the proof for $A = [0,p] \nsubseteq C$ and $B = [0,d] \subset \mathbb{R}$, $d > 0$. We let

$$L = C \times (-\infty,0] \, , \quad G = [0,p] \times [0,d] \; , \tag{4.24}$$

$$H = [p,0] \times [0,d] \; , \quad K = C \times [d,\infty) \; .$$

For any collection $\{Z(t): t \in C \times \mathbb{R}\}$ and any set $S \subset C \times \mathbb{R}$, we write Z_S for the σ-algebra generated by $\{Z(t): t \in S\}$. Similarly, we write M_S for the σ-algebra generated by $\{Mf: f = 1_U, U \subset S\}$. Finally, if $\mathbb{G}$, $\mathbb{H}$, $\mathbb{K}$ are sub-σ-algebras of $\mathbb{F}$, we write (read $\mathbb{H}$ splits $\mathbb{G}$ and $\mathbb{K}$)

$$\mathbb{G} \,]\, \mathbb{H} \,[\, \mathbb{K}$$

to mean "$\mathbb{G}$ and $\mathbb{K}$ are conditionally independent given $\mathbb{H}$." In the special case when $\mathbb{G}$ and $\mathbb{K}$ are independent, we write

$$\mathbb{G} \,][\, \mathbb{K} .$$

The following two elementary facts will be used a number of times:

$$(4.25) \quad \mathbb{G} \vee \mathbb{B} \,]\, \mathbb{H} \,[\, \mathbb{K} \Rightarrow \mathbb{G} \vee \mathbb{B} \,]\, \mathbb{H} \vee \mathbb{B} \,[\, \mathbb{K} \Rightarrow \mathbb{G} \vee \mathbb{B} \,]\, \mathbb{H} \vee \mathbb{B} \,[\, \mathbb{K} \vee \mathbb{B}$$

$$(4.26) \quad \mathbb{G} \,]\, \mathbb{H} \,[\, \mathbb{K}, \quad \mathbb{L} \,][\, \mathbb{G} \vee \mathbb{H} \vee \mathbb{K} \Rightarrow \mathbb{G} \,]\, \mathbb{H} \,[\, \mathbb{K} \vee \mathbb{L} .$$

We start by letting

$$(4.27) \qquad Y(p,t) = \int_{C \times (0,t]} e^{-am(p,q)-bm(s,t)} M(dq,ds) .$$

Applying the arguments of the proof of Theorem (2.2), Markov property, to the process $p \longrightarrow Y(p,\cdot)$, which has exactly the same form as the process $p \longrightarrow X_p$ of (2.2), we see that

(4.28) $$Y_G \;]\; Y_{G\cap H} \;[\; Y_H \,.$$

Since $G \supset G \cap K$, (4.25) and (4.28) yield

(4.29) $$Y_G = Y_G \vee Y_{G\cap K} \;]\; Y_{G\cap H} \vee Y_{G\cap K} \;[\; Y_H \vee Y_{G\cap K} \,.$$

By the independence of the increments of M over disjoint sets, we have

$$M_K \;][\; Y_G \vee Y_H \,,$$

which yields, when used with (4.26) and (4.29),

(4.30) $$Y_G \;]\; Y_{G\cap H} \vee Y_{G\cap K} \;[\; Y_H \vee Y_{G\cap K} \vee M_K = Y_H \vee Y_K \,.$$

Note that the values of Y on $G \cup H \cup K$ are determined by M on that set, whereas the values of X on L are determined by M on L. So,

$$X_L \;][\; Y_G \vee Y_H \vee Y_K \,,$$

which together with (4.30) and (4.26) implies that

(4.31) $$Y_G \;]\; Y_{G\cap H} \vee Y_{G\cap K} \;[\; Y_H \vee Y_K \vee X_L \,.$$

Since $X_L = X_L \vee X_{G\cap L}$, we may put $X_{G\cap L}$ into every term in (4.31) by using (4.25), which gives

$$(4.32) \qquad Y_G \vee X_{G\cap L} \;]\; Y_{G\cap H} \vee Y_{G\cap K} \vee X_{G\cap L} \;[\; Y_H \vee Y_K \vee X_L \,.$$

It follows from (4.27) defining Y and (4.14) for X that $X(p,t) = e^{-bt} X(p,0) + Y(p,t)$. Thus, X on $G\cap L$ together with Y on G determine X on G, that is,

$$(4.33) \qquad Y_G \vee X_{G\cap L} = X_G \,.$$

Putting (4.33) into (4.32), noting that $X_G = X_G \vee X_{G\cap K}$, and applying (4.25) once again, we obtain

$$(4.34) \quad X_G \;]\; Y_{G\cap H} \vee Y_{G\cap K} \vee X_{G\cap L} \vee X_{G\cap K} \;[Y_H \vee Y_K \vee X_L \vee X_{G\cap K} \,.$$

Finally, noting that

$$Y_{G\cap H} \vee Y_{G\cap K} \vee X_{G\cap L} \vee X_{G\cap K} = X_{G\cap L} \vee X_{G\cap K} \vee X_{G\cap K} = X_{\partial G}$$

and that

$$Y_H \vee Y_K \vee X_L \vee X_{G\cap K} = X_L \vee X_H \vee X_K$$

we see that (4.34) is in fact

$$X_G \;]\; X_{\partial G} \;[\; X_{L\cup H\cup K} \,,$$

which is the desired statement of the Markov property.

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